MTH 301: Group Theory

Homework VII

(Due 24/10)

Problems for submission

1. Establish the assertions in 5.1 (iii), 5.1(viii), 5.2(ii), and 5.2 (v) of the Lesson Plan.

2. Let G be a nontrivial group. Two elements $g, h \in G$ are said to be *conjugate in* G if there exists $x \in G$ such that $g = xhx^{-1}$. Now define a relation $\sim_c$ on G by

$$g \sim_c h \iff g \text{ and } h \text{ are conjugate.}$$

   Show that $\sim_c$ defines an equivalence relation on G.

   Each equivalence class (denoted by $[g]_c$) induced by the relation $\sim_c$ is called a *conjugacy class of* G.

3. Consider the partition of S_n into distinct conjugacy classes under the equivalence relation $\sim_c$ mentioned above.

   (a) If $\sigma \in S_n$ is an m-cycle and $\sigma' \in S_m$ is an ℓ-cycle, then show that

$$[\sigma]_c = [\sigma']_c \iff m = \ell.$$

   [Hint: Start by showing that given a permutation σ and a cycle $(i_1\, i_2\, \ldots\, i_k)$, $\sigma(i_1\, i_2\, \ldots\, i_k)\sigma^{-1} = (\sigma(i_1)\, \sigma(i_2)\, \ldots\, \sigma(i_k))$.]

   (b) Suppose that the unique cycle decomposition of a permutation $\sigma \in S_n$ is given by

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_{k_\sigma},$$

   where each σ_i is an m_i-cycle. Then, as $\displaystyle\sum_{i=1}^{k_\sigma} m_i = n$, this decomposition induces a *partition* P_σ of the integer n. (In Number Theory, a *partition of a positive integer* n is a way of writing n as a sum of positive integers, up to reordering of summands.) Then show that:

   (i) $o(\sigma) = \text{lcm}(m_1, m_2, \ldots, m_{k_\sigma})$.

   (ii) Given two permutations $\sigma_1, \sigma_2 \in S_n$,

$$[\sigma_1]_c = [\sigma_2]_c \iff P_{\sigma_1} = P_{\sigma_2}.$$

   (c) Using (b), determine the number of distinct conjugacy classes of S_n.

Problems for practice

1. Establish the assertion of uniqueness in 5.2 (vi) of the Lesson Plan.
2. Establish the assertion in 5.3 (iii) of the Lesson Plan.
3. Show that for $n \geq 3$, there exists a momomorphism $D_{2n} \to S_n$.
4. Show that for $n \geq 3$, there exists a momomorphism $S_n \to S_{n+1}$.
5. Show that every normal subgroup of S_n is a disjoint union of conjugacy classes.
6. For $n \geq 3$, show that the following sets of transpositions generate S_n.

   (a) The set $A = \{(i\, i+1) : 1 \leq i \leq n-1\}$.

   (b) For $1 \leq j \leq n$, the set $B_j = \{(j\, i) : 1 \leq i \leq n \text{ and } j \neq i\}$.

   [Hint: Start by showing that every other transposition (and hence every k-cycle) is a product of the transpositions in either A or the B_j. Then use the unique cycle decomposition property of a permutation.]